Stochastic resonant media: Signal-to-noise ratio for the activator-inhibitor system through a quasivariational approach

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We have made an analytical study of the phenomenon of stochastic resonance in a spatially extended stochastic system of the activator-inhibitor kind. In its bistable regime, through a *quasivariational* approach we make an approximate evaluation of the *nonequilibrium potential* for this system. The latter in turn allows us to obtain the probability for the decay of the (extended) metastable states and through it the signal-to-noise ratio within the framework of a two-state description. The analytical results show that this ratio increases with the activator's diffusivity, whereas it exhibits nonmonotonic behavior against variation of the coupling between both fields. [S1063-651X(98)09605-6]

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In recent years the phenomenon of *stochastic resonance* (SR) has attracted considerable attention and has been extensively studied from both the theoretical and experimental points of view [1]. SR is the name coined for the rather counterintuitive fact that the response of a *nonlinear* system to a periodic signal may be *enhanced* with the addition of an optimal amount of noise. The key parameter here is the *signal-to-noise ratio* (SNR) at the output. Very recently, attention has been focused on the occurrence of this phenomenon in coupled or *extended* systems [2–4], which we might call *stochastic resonant media* (SRM) [4]. A very convenient tool (where available) for the analysis of SRM is provided by their *nonequilibrium potential* (NEP) [5] since it allows for a straightforward computation of transition probabilities between attractors.

In previous works [4] (with the aim to encourage experimentalists dealing with distributed electronic, chemical, or biological systems to search for alternative variables to tune up so as to enhance the stochastic resonant response of the system) we have exploited our own results in finding a NEP for several reaction-diffusion (RD) models [6] to draw conclusions on the dependence of the SNR upon several relevant parameters. Among them, we have studied an effective one-component RD model that is equivalent to a two-component one of the activator-inhibitor type, in the limit of fast inhibition [7,8]. A response enhancement was observed not only as a result of an increase in the (local) diffusive coupling, but also due to an increase of a *nonlocal* effective self-coupling that arises in this limit as a reminder of the coupling β between the activator and inhibitor fields. In this paper we venture beyond the fast-inhibitor limit and analyze the SR phenomenon in the *full* activator-inhibitor system. We study the bistable regime, where, in the phenomenologically relevant case of a very slow inhibitor, a *quasivariational* approximation can be used to obtain the NEP through a singular perturbation approach [9]. The latter is exploited in turn to obtain the SNR when the system is subjected to a modulated weak signal, within the framework of a two-state description [10]. In agreement with previous studies [2,4], our present results indicate that this ratio increases with the diffusivity (or local coupling parameter). However, at variance with the fast-inhibitor case [8], it exhibits a nonmonotonic response to variations in the parameter that couples both fields.

The system we consider here is a one-dimensional and piecewise-linear realization of the ''activator-inhibitor'' class of RD models:

$$\frac{\partial u(x,t)}{\partial t} = (\epsilon\tau_r)^{-1}\left\{ \epsilon^2 \frac{\partial^2 u}{\partial x^2} - u + \theta(u-a) - v \right\},$$

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \beta u - \gamma v. \tag{1}$$

This simple two-component RD model, while keeping the essential features of the bistable and excitable regimes of the original models, is at the same time amenable to analytical calculations. Under the names of Fitzhugh-Nagumo and Bonhoffer–van der Pol models it has been used to model a diversity of excitable systems, allowing for the theoretical prediction of stationary patterns, which later were observed in experiments [11,12]. Systems of this kind always have a ''rest'' state, i.e., a (locally) stable steady state with uniform concentration, so the fields $u(x,t)$ and $v(x,t)$ represent the local differences between the *activator* and the *inhibitor* concentrations and their respective rest values. The constants ϵ, τ_r, β, and γ are all positive and $\theta(x)$ is the Heaviside step function. Insofar as the ''excitability threshold'' a is positive, the piecewise-linear dynamics is essentially the same as the original nonlinear one. The bistable regime of the system corresponds to $a<\gamma/(\beta+\gamma)$ in Eq. (1). For $\beta>0$ the set of equations (1) is *nonvariational* because the standard conditions needed to define a potential are not fulfilled [5]. When the inhibitor field v follows the activator *adiabatically* (that is, when $\epsilon\tau_r\to\infty$ and one can assume $\partial v/\partial t=0$) the system reduces to a single time-evolution equation (for the activator

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field) with a *nonlocal* contribution in the space coordinate and it is possible to write out a NEP for this case [7]. In this work we shall investigate instead the case $\epsilon \ll 1$, where a ''quasivariational'' approximation is available [9].

The system described by Eqs. (1) is assumed to be defined in an interval of length $2L$, with Dirichlet boundary conditions. The stationary solution is written as a linear combination of exponentials $\exp(k_i x)$, where the k_i are the roots of the characteristic polynomial [12]. The solutions under consideration are such that a central part of the system forms a frozen ''activated'' region, i.e., $u(x) > a$. The solution is piecewise analytic with matching points at $|x| = x_c$, defined by $u(x_c) = a$. The coefficients are fixed by boundary and matching conditions. The stationary solutions are independent of τ_r since this is a *dynamical* parameter that specifies the ratio between the relaxation constant of u and that of v.

When the parameter ϵ is small enough, the stationary concentration patterns have two regions where both the activator and inhibitor fields vary in a smooth fashion, separated by a sharp interface or *boundary layer* (BL) where the activator field u varies abruptly [9,11]. In this case a quasivariational approximation is possible, which allows one to identify the globally stable stationary solution of Eqs. (1). Following Ref. [9], where such a method was developed, we divide the coordinate space into two parts. One is the shape of u within the boundary layer, where we exploit the fact that the variation of the field v is not significant. The NEP is approximated in this region as

$$\mathcal{L}_{\mathrm{BL}}\{u(x), v(x) \sim v_I\} = \frac{1}{\epsilon \tau_r} \int_{X_c - \epsilon}^{X_c + \epsilon} dx \left\{ \frac{\epsilon^2}{2} \left(\frac{du}{dx} \right)^2 + \frac{u^2}{2} + v_I u - (u - a)\, \theta(u - a) \right\}, \tag{2}$$

where v_I is a characteristic value of the inhibitor field at the interface [in our calculation $v_I = v(x_c)$]. The integration domain is limited to the region where $|x_c| - \epsilon \leqslant |x| \leqslant |x_c| + \epsilon$. Outside this boundary layer, we approximate the stationary behavior of the activator by using the fact that $\epsilon \sim 0$. This approximation is the basis of the singular perturbation method [11]. In that case $v = -u + \theta(u - a) \rightarrow u = h^{\pm}(v)$ and the NEP in these regions results:

$$\mathcal{L}_{\mathrm{outer}}\{u(x) \sim h^{\pm}(v), v(x)\} = \int_{(\mathrm{outer})^{\pm}} dx \left\{ \frac{1}{2} \left(\frac{dv}{dx} \right)^2 + \frac{1 + \gamma}{2} v^2 + (v_I - v)\, \theta(x_c - |x|) \right\}. \tag{3}$$

Here the integration domain is limited to the regions $(\mathrm{outer})^{+} = \{|x| \leqslant |x_c| - \epsilon\}$ and $(\mathrm{outer})^{-} = \{|x_c| + \epsilon \leqslant |x| \leqslant L\}$.

Hence, for $\epsilon \ll 1$ the NEP for Eqs. (1) can be approximated *in the stationary case* as

$$\mathcal{L}_{\mathrm{NE}} = \mathcal{L}_{(\mathrm{outer})^{+}} + \mathcal{L}_{(\mathrm{outer})^{-}} + \mathcal{L}_{BL}\,. \tag{4}$$

The locally stable uniform state $u = v = 0$ can also be included in Eq. (4) NEP. As the values of u, v, and v_I are zero at this uniform state, we find $\mathcal{L}_{\mathrm{NE}}\{u = 0, v = 0\} = 0$.

We can now analyze the global stability of the resulting stationary patterns distinguishing the stable from the metastable ones through the NEP given in Eq. (4) and calculate the height of the barrier between these attractors. The linear analysis indicates that there are two stable stationary solutions: the homogeneous one $u_0 = v_0 = 0$ and the nonhomogeneous solution with the largest activated zone (u_s, v_s). Figure 1 shows, in agreement with previous studies [7,8], the dependence of the potential with the threshold parameter a in the neighborhood of the bistable regime, where $a = a_c$. Clearly, the range of ϵ in this analysis is limited to small values in order that the singular perturbation scheme be valid.

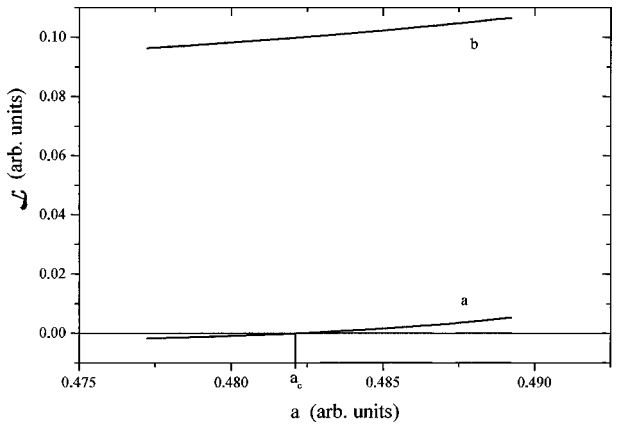

FIG. 1. Nonequilibrium potential $\mathcal{L}$ (evaluated at the stationary patterns) as a function of a for $\epsilon = 0.005$ and $L = 1$. Bottom curve, *stable* nonhomogeneous solution; top curve, *unstable* one. The bistability point $a = a_c$ is indicated.

As in previous studies [4,8], we shall exploit a scheme, based on path-integral techniques, that allows one to describe the decay of extended metastable states in a bistable situation (only two attractors) [13], yielding the following Kramers-like result for the decay time:

$$\langle \tau \rangle = \tau_0 \exp\left\{ \frac{\Delta \mathcal{L}_{\mathrm{NE}}[u, v]}{\eta} \right\}, \tag{5}$$

with $\Delta \mathcal{L}_{\mathrm{NE}}[u, v] = \mathcal{L}_{\mathrm{NE}}[u_u(y), v_u(y)] - \mathcal{L}_{\mathrm{NE}}[u_m(y), v_m(y)]$. The prefactor τ_0 is determined by the curvature of $\mathcal{L}_{\mathrm{NE}}[u, v]$ at its extrema and η is the intensity of the white noise, which is assumed to enter additively into the activator equation.

In order to analyze the SR phenomenon in this spatially extended system we subject the system, in the indicated bistable regime, to an external modulated weak signal of the form $a(t) = a_c + \delta a_c \cos(\Omega t + \varphi)$, where a_c is the threshold value at which $\mathcal{L}_{\mathrm{NE}}[u_s(y), v_s(y)] \equiv 0$. We shall follow a procedure already described in Ref. [8] and based on the two-state approximation of McNamara and Wiesenfeld [10]. If the frequency Ω of the external periodic modulation is small compared to other inverse time scales (such as the unperturbed Kramers rate), the stochastic stationarity can be continuously achieved (i.e., the probability density adjusts adiabatically to the changing nonequilibrium potential). In this adiabatic limit, the stationary Fokker-Planck dynamics can be used to compute probability densities and correlation functions. In the bistable situation we need to concentrate only on the transitions between the two stable stationary states $u_0 = v_0 = 0$ and u_s, v_s through the saddle defined by u_u, v_u. We hence calculate the correlation function between

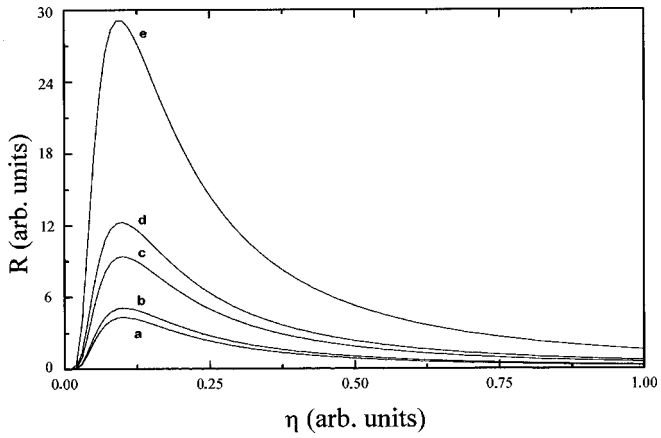

FIG. 2. SNR for the activator-inhibitor model vs the noise intensity γ for some values of ϵ: (a) 0.48, (b) 0.49, (c) 0.5, and (d) 0.51. We have set $\beta=0.9$, $L=1$, $a=a_c$, and $\delta a_c=0.01$.

these two states through the evaluation of the transition probabilities between them, $W_{0,s}=\tau_0^{-1}\exp(-\Delta\mathcal{L}_{\rm NE}^{0,s}[u_0,v_0;u_s,v_s]/\eta)$, which appear in the associated master equation. τ_0 is given by the asymptotically dominant linear stability eigenvalues $\tau_0=2\pi/\sqrt{|\lambda^u|\overline{\lambda^s}}$ (λ^u is the only unstable eigenvalue around u_u,v_u and $\overline{\lambda^s}$ is the *average* of the smallest eigenvalues around $u_0=v_0=0$ and u_s,v_s) [8]. Since for small δa_c

$$\Delta\mathcal{L}_{\rm NE}^{0,s}\approx\Delta\mathcal{L}_{\rm NE}[u_0,v_0;u_s,v_s]+\delta a_c\left[\frac{\partial\Delta\mathcal{L}_{\rm NE}^{0,s}[\phi,\phi_c]}{\partial a}\right]_{a_c}\cos(\Omega t+\varphi),\qquad(6)$$

we can evaluate the correlation function by solving the master equation up to first order in δa_c and, by performing its Fourier transform in time, get the power spectral density $S(\omega)$. Using a by now standard definition of the SNR [10], we finally obtain its relevant part, given by

$$R\sim\left(\frac{\Lambda}{\tau_0\eta}\right)^2\exp(-2\Delta\mathcal{L}_{\rm NE}[u,v]/\eta),\qquad(7)$$

with $\Lambda=[\partial\Delta\mathcal{L}_{\rm NE}/\partial a]_{a_c}\delta\phi_c$. Equation (7) is similar in form to that of zero-dimensional systems, but here Λ, τ_0, and

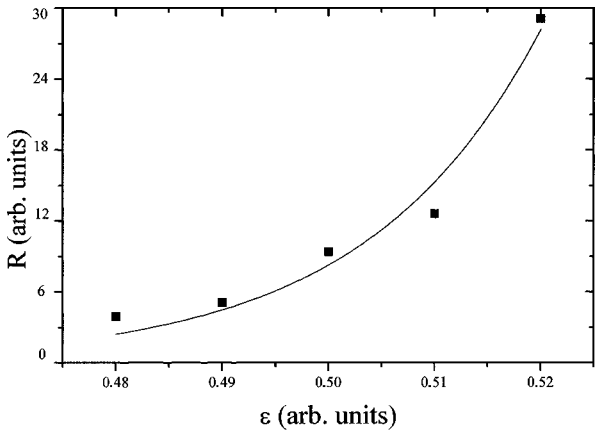

FIG. 3. Maximum SNR as a function of ϵ for the same parameters as in Fig. 2. The line corresponds to a powerlike fit of the data.

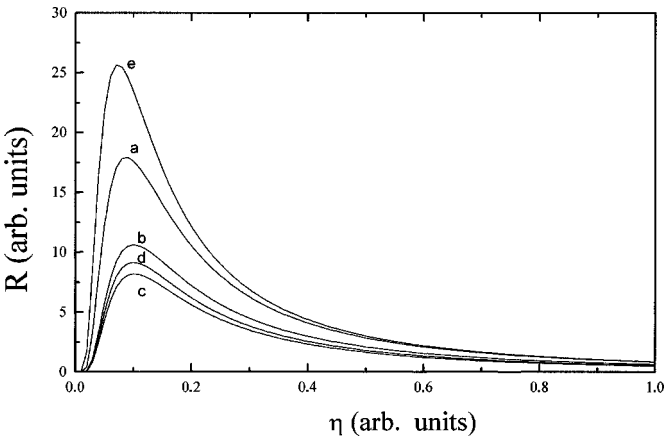

FIG. 4. SNR as a function of γ for different values of β: (a) 0.7, (b) 0.9, (c) 1.0, (d) 1.1, and (e) 1.5. We have set $\epsilon=0.005$ and the other parameters as in Fig. 3.

$\Delta\mathcal{L}_{\rm NE}[u,v]$ contain all the relevant information regarding the spatially extended character of the system.

Figure 2 shows the dependence of the SNR on the noise intensity η for a few values of ϵ, while Fig. 3 shows that of the maximum SNR on ϵ (always for fixed β). The enhancement of the SNR with increasing ϵ is apparent. As in the one-component model [2,4], such an increase shows a powerlike dependence, at least in the limited range of values of ϵ we have analyzed, as indicated by the continuous line in Fig. 3. Figure 4 shows the dependence of the SNR on η for a few values of β and Fig. 5 that of the maximum SNR on β for fixed ϵ. At variance with the adiabatic case [8], there is a nonmonotonic behavior in the system's response against variation of β. As indicated in Fig. 5, this behavior is adequately fitted by a parabolic function.

Summarizing, we have studied the phenomenon of SR in a coupled or extended system (or SRM) with the aim of recognizing parameters other than η that could be tuned up for an easier detection of the phenomenon. Particularly, we have analyzed the bistable and $\epsilon\ll1$ regime of a piecewise-linear activator-inhibitor model, where a quasivariational approximation allows us to obtain the nonequilibrium potential. This effective potential was exploited to obtain, within the

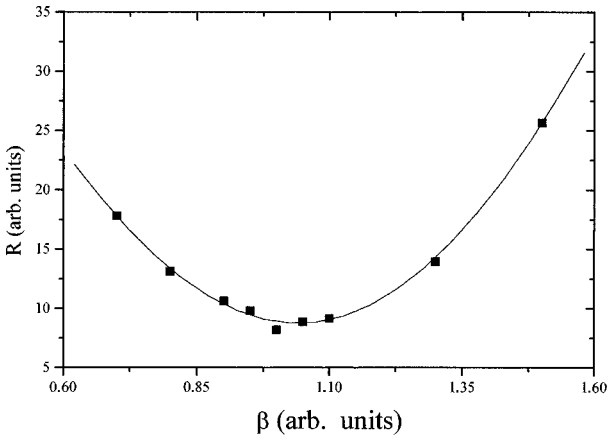

FIG. 5. Maximum SNR as a function of β for $\epsilon=0.005$ and the other parameters as in Fig. 4. The line corresponds to a parabolic fit of the data.

framework of a two-state description, the signal-to-noise ratio for this system. The present results show, in agreement with those obtained for the one-component model, a power-like increase of the SNR as a function of the diffusion constant ϵ that plays the role of the spatial coupling parameter. However, at variance with the fast-inhibitor case studied before [8], they also show a nonmonotonic behavior of the system's response against variation of the parameter β that couples both fields. The clarification of the role played by this latter parameter will require further study, for instance, analyzing the effect of the indicated parameter alone by neglecting the spatial or diffusive coupling. This is part of a work under way [14].

The relevance of the present results for technological applications in signal detection as well as their biological implications are apparent. Many distributed electronic circuits can be regarded in the continuum limit as a set of diffusively coupled nonlinear oscillators. With regard to chemical systems, particularly interesting results ensue from recent experiments on several reactions done under good-stirring conditions [15] (corresponding to transitions between a focus and an oscillatory state via a Hopf bifurcation, where both are homogeneous states) as well as a more recent and also tightly related experimental result that corresponds to the case of resonant pattern formation in a chemical system [16], indicating the possibility of the appearance of SR under non-stirring conditions. Even though such cases cannot be described by the activator-inhibitor model in the above-indicated limit, they make apparent the relevance of such results and the interest of further studies exploiting the approach shown here. Since the present results predict a strong dependence of the SR upon both spatial and interspecies coupling parameters, we hope that they can motivate not only the simulation of coupled sets of such nonlinear oscillators but also the experimental search of this spatially dependent phenomenon in chemical and coupled electronic systems. In particular, we expect that by exploiting an experimental setup similar to the one in Ref. [16], with a low-amplitude (below-threshold) forcing plus noise, a SRM phenomenon will show up.

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